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A Fast Maximum Likelihood Estimation and Detection Algorithm for Bernoulli–Gaussian Processes

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Abstract—**In this correspondence, we propose a fast maximum likelihood detection and estimation algorithm, called a multiple-most-likely-replacement (MMLR) detector, for Bernoulli–Gaussian processes which are distorted by a linear time-invariant system and contaminated by a white Gaussian noise. This new detector works as well as the well-known single-most-likely-replacement (SMLR) detector. However, the former is computationally faster than the latter. We discuss two examples which demonstrate the computational advantage of the proposed algorithm using synthetic data.**

I. INTRODUCTION

A randomly located spike train $\mu(k)$ can be modeled as the following product model:

$$\mu(k) = r(k)\, q(k) \tag{1}$$

where $q(k)$ is a binary sequence with value zero or unity and $r(k)$ depicts spike amplitudes. $q(k) = 1$ indicates that a spike is located at the time point k with amplitude $r(k)$. On the other hand, $q(k) = 0$ indicates that there is no spike located at the time point k. This product model has been applied in modeling the reflectivity sequence in reflection seismology [1], [2], [4].

Kormylo and Mendel [3] developed an iterative single-most-likely-replacement (SMLR) detector for estimating $\mu(k)$ using a given set of measurements $z(k)$, $k = 1, 2, \cdots, N$ where

$$z(k) = \mu(k) * v(k) + n(k) = \sum_{i=0}^{k} v(i)\, \mu(k-i) + n(k) \tag{2}$$

in which $n(k)$ is the measurement noise; $v(k)$, $k = 0, 1, 2, \cdots$, is the impulse response sequence of the signal distorting system (e.g., impulse response of communication channel, seismic source wavelet); and $\mu(k)$ is the desired signal sequence (e.g., message, reflectivity sequence) which is modeled as (1). The statistical assumptions about $r(k)$, $q(k)$, and $n(k)$ used by them are the following.

(A1) $r(k)$ is white zero-mean Gaussian with variance C.

(A2) $n(k)$ is white zero-mean Gaussian with variance R.

(A3) $q(k)$ is a Bernoulli process of zeros and ones with the probability mass function

$$P_r[q(k) = q] = \begin{cases} \lambda, & \text{if } q = 1 \\ 1 - \lambda, & \text{if } q = 0. \end{cases} \tag{3}$$

(A4) $r(k)$, $n(k)$, and $q(k)$ are independent.

The SMLR detection algorithm was developed by maximizing the following likelihood function:

$$S\{\boldsymbol{q}|z\} = p(z|\boldsymbol{q})\, P_r(\boldsymbol{q}) \tag{4}$$

where $\boldsymbol{q} = \text{col}\,(q(1), q(2), \cdots, q(N))$, and $z = \text{col}\,(z(1), z(2), \cdots, z(N))$.

Let $q_t(q_r(k), i)$ be a test sequence associated with a reference sequence $q_r(i)$:

$$q_t(q_r(k), i) = \begin{cases} 1 - q_r(i), & i = k \\ q_r(i), & \text{otherwise.} \end{cases} \tag{5}$$

Note that k in $q_t(q_r(k), i)$ is a parameter instead of a time point. Let

$$\boldsymbol{q}_t(q_r(k)) = \text{col}\,\big(q_t(q_r(k), 1), q_t(q_r(k), 2), \cdots, q_t(q_r(k), N)\big). \tag{6}$$

The likelihood ratio of $\boldsymbol{q}_t = \boldsymbol{q}_t(q_r(k))$ to $\boldsymbol{q}_r$ is defined as

$$\Lambda_{tr}(k, \boldsymbol{q}_r) = \frac{S\{\boldsymbol{q}_t(q_r(k))|z\}}{S\{\boldsymbol{q}_r|z\}}. \tag{7}$$

Note that $\Lambda_{tr}(k, \boldsymbol{q}_r)$ is the likelihood ratio of $\boldsymbol{q}_t$ to $\boldsymbol{q}_r$ where $\boldsymbol{q}_t$ differs from $\boldsymbol{q}_r$ only at the time point k. Let k^* be associated with the maximum value of $\Lambda_{tr}(k, \boldsymbol{q}_r) > 1$ where $k \in \{1, 2, \cdots, N\}$. Then the single-most-likely-replacement test sequence is $\boldsymbol{q}_t^* = \boldsymbol{q}_t(q_r(k^*))$, and the likelihood function evaluated at $\boldsymbol{q}_t^*$ is at least as large as its value evaluated at $\boldsymbol{q}_r$.

The SMLR search algorithm, initiated by $\boldsymbol{q}_r = \boldsymbol{q}^{(0)}$, computes N log-likelihood ratios corresponding to N different $\boldsymbol{q}_t$ sequences $(\boldsymbol{q}_t(q_r(k)), k = 1, 2, \cdots, N)$. The most likely $\boldsymbol{q}_t$ sequence is used as the reference sequence $\boldsymbol{q}^{(1)}$. If, after i iterations, we obtain a reference $\boldsymbol{q}_r = \boldsymbol{q}^{(i)}$ which is more likely than any of the corresponding $\boldsymbol{q}_t$ sequences, then the search stops and $\boldsymbol{q}^{(i)}$ is the final detected sequence. The SMLR detector has been successfully applied in seismic deconvolution by Chi *et al.* [2]. There are some other existing detection algorithms such as threshold detector [4] and Viterbi algorithm detector [5] for detecting $q(k)$ and estimating $r(k)$. These two detectors are noniterative and so their computational load is constant. As mentioned in [4], the threshold detector is optimal when wavelets are nonoverlapping. When wavelets are severely overlapped, its performance is worse than the SMLR detector. The Viterbi algorithm detector has a very nice parallel processing structure and its performance is comparable to the SMLR detector. The reader is referred to [4] and [5] for details of these two detectors.

Although the SMLR detector works well, it requires a fixed-interval optimal smoother [3], [4] for computing $\Lambda_{tr}(k, \boldsymbol{q}_r)$ for k

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$= 1, 2, \cdots, N$ to add or to remove only a single spike at each iteration. A considerably large computational effort is required in order to complete the procedure. In this correspondence, we propose a new fast detection algorithm based on the same statistical assumptions and likelihood function. The proposed detector, which is called a multiple-most-likely-replacement (MMLR) detector, works as well as the SMLR detector. The MMLR detection algorithm is structurally the same as the SMLR detection algorithm. The only difference between these two detection algorithms is the approach of updating spike locations at each iteration. The MMLR detection algorithm requires the same computational effort (an optimal smoother) as the SMLR detection algorithm at each iteration. However, it adds or removes $l \geq 1$ spikes at each iteration where l varies from iteration to iteration. Therefore, the MMLR detector is computationally faster than the SMLR detector; this multiple replacement permits a greater increase in the likelihood function at each iteration. Both the MMLR and SMLR detectors guarantee that the likelihood function increases at each iteration. After the detection of $q(k)$ is completed, the estimation of $r(k)$ can be done by applying Mendel's minimum variance deconvolution (MVD) filter [1], [4], [7].

When not only $q(k)$ and $r(k)$ but also other parameters such as $v(k)$ and statistical parameters associated with $q(k)$ and $r(k)$ are unknown, the maximum likelihood deconvolution (MLD) algorithm [1], [2], [4] can be used to estimate all unknown quantities simultaneously.

In Section II, we develop the MMLR detection algorithm. In Section III, we show why the MMLR detection algorithm works and discuss its properties. In Section IV, we discuss two simulation examples which show the computational advantage of the MMLR detection algorithm over the SMLR detection algorithm. We summarize our results and draw conclusions in the final section.

II. MMLR Detection Algorithm

Assume that $v(k) = 0$ for $k \notin [0, M)$ and that we are given a reference sequence $\boldsymbol{q}_r$. Let A_i be the integer set

$$A_i \triangleq A_{i-1} \cup (n_{1i} - M, n_{2i} + M), \qquad i \geq 1 \tag{8}$$

and let k_i be the time point where

$$\Lambda_{tr}(k_i, \boldsymbol{q}_r) = \max \left\{ \Lambda_{tr}(k, \boldsymbol{q}_r) \,|\, k \in B_{i-1} \right\} > 1 \tag{9}$$

where B_i is the integer set

$$B_i \triangleq \left\{ k \,|\, k \in B_0,\ k \notin A_i \right\} \tag{10}$$

with $B_0 = [1, N]$, $A_0 = \varnothing$, and $n_{1i} \leq k_i$ is the largest integer such that $q_r(k) = 0$ for $k \in (n_{1i} - M, n_{1i})$ and $n_{2i} \geq k_i$ is the smallest integer such that $q_r(k) = 0$ for $k \in (n_{2i}, n_{2i} + M)$. If n_{1i} does not exist, we set $n_{1i} = 1$. If n_{2i} does not exist, we set $n_{2i} = N$. Note that $A_i \subseteq A_{i+1}$ and $A_i \cap B_i = \varnothing$ for all i.

By searching for $(k_i, n_{1i}, n_{2i}, A_i, B_i)$, $i = 1, 2, \cdots, l$ recursively, the MMLR detected sequence $q^*(k)$ is given by

$$q^*(k) = \begin{cases} 1 - q_r(k), & k \in \{k_1, \cdots, k_l\} \\ q_r(k), & \text{otherwise} \end{cases} \tag{11}$$

where l is the smallest integer such that either B_l is an empty set or $\Lambda_{tr}(k, \boldsymbol{q}_r) \leq 1\ \forall k \in B_l$. In other words, this recursive search for k_i stops when this smallest l is found. Note that $q_r(k) = 0$ for $k \in \Gamma_i$, $1 \leq i \leq l$, where

$$\Gamma_i = \left\{ \boldsymbol{q} \,|\, q(k) = 0, \quad k \in (n_{1i} - M, n_{1i}) \cup (n_{2i}, n_{2i} + M) \right\}. \tag{12}$$

The sequence $\boldsymbol{q}^*$ serves as the reference sequence for the next iteration of the MMLR detector. Observe that $\boldsymbol{q}_r$ and $\boldsymbol{q}^*$ differ in l locations, and l can be different at each iteration. This procedure is iterated until $\Lambda_{tr}(k, \boldsymbol{q}_r) \leq 1$ for all $1 \leq k \leq N$. At each iteration, the likelihood ratio for $\boldsymbol{q}_t = \boldsymbol{q}^*$ with respect to $\boldsymbol{q}_r$ can be shown to be (see below)

$$\Lambda_{tr}^* = \frac{S\{\boldsymbol{q}^* | \boldsymbol{z}\}}{S\{\boldsymbol{q}_r | \boldsymbol{z}\}} = \prod_{i=1}^{l} \Lambda_{tr}(k_i, \boldsymbol{q}_r) > 1 \tag{13}$$

which is shown in the next section. Therefore, we guarantee that S increases at each iteration.

III. Analysis of the MMLR Detector

In this section, we analyze the MMLR detection algorithm. Five properties of the MMLR detector are presented in this analysis. We need the following theorem.

Theorem 1: Assume that $v(k) = 0$ for $k \notin [0, M)$. Let D be the integer set $D = (n_1 - M, n_2 + M)$ and

$$\Gamma = \left\{ \boldsymbol{q} \,|\, q(k) = 0, \ k \in (n_1 - M, n_1) \cup (n_2, n_2 + M) \right\} \tag{14}$$

where $n_1 \leq n_2$. If two reference sequences $q_r^{(1)}(k) \in \Gamma$ and $q_r^{(2)}(k) \in \Gamma$, differ in only one location $m \in [n_1, n_2]$, then the associated likelihood ratios [see (5) and (7)] satisfy

$$\Lambda_{tr}(k, \boldsymbol{q}_r^{(1)}) = \Lambda_{tr}(k, \boldsymbol{q}_r^{(2)})\ \forall k \notin D. \tag{15}$$

The proof is given in Appendix A.

Based on this theorem, we can show the following properties of the MMLR detector.

Property 1: Assume that $q_r(k)$ is the reference sequence and $q^*(k)$ given by (11) is the detected sequence at a certain iteration. Then the likelihood ratio, Λ_{tr}^*, comparing $\boldsymbol{q}^*$ to $\boldsymbol{q}_r$, is given by (13).

Proof: Let

$$q_t^{(i)}(k) = \begin{cases} 1 - q_r(k), & k \in \{k_1, \cdots, k_i\} \\ q_r(k), & \text{otherwise} \end{cases} \tag{16}$$

for $i \geq 1$ and $q_t^{(0)}(k) = q_r(k)$. Note that $q^*(k) = q_t^{(l)}(k)$. One can also see that $q_t^{(i)}(k) \in \Gamma_i$ and $q_t^{(i-1)}(k) \in \Gamma_i$ [see (12)] differ in only one location $k = k_i \in [n_{1i}, n_{2i}] \subset (n_{1i} - M, n_{2i} + M) \subset A_i$ [see (8)] and $A_i \cap B_i = \phi$ [see (10)]. By Theorem 1 we see that

$$\Lambda_{tr}(k, \boldsymbol{q}_t^{(i)}) = \Lambda_{tr}(k, \boldsymbol{q}_t^{(i-1)})\ \forall k \in B_i. \tag{17}$$

Note that $\Lambda_{tr}(k, \boldsymbol{q}_t^{(0)}) = \Lambda_{tr}(k, \boldsymbol{q}_r)$ for all $k \in B_0$. Therefore,

$$\Lambda_{tr}(k, \boldsymbol{q}_t^{(i)}) = \Lambda_{tr}(k, \boldsymbol{q}_r)\ \forall k \in B_i,\ \forall i \geq 1. \tag{18}$$

Because $k_i \in B_{i-1}$, we have [see (9)]

$$\begin{aligned} S\{\boldsymbol{q}_t^{(i)}|\boldsymbol{z}\}/S\{\boldsymbol{q}_t^{(i-1)}|\boldsymbol{z}\} &= \Lambda_{tr}(k_i, \boldsymbol{q}_t^{(i-1)}) \\ &= \Lambda_{tr}(k_i, \boldsymbol{q}_r) > 1. \end{aligned} \tag{19}$$

Therefore, the likelihood ratio $S\{\boldsymbol{q}^*|\boldsymbol{z}\}/S\{\boldsymbol{q}_r|\boldsymbol{z}\}$ is

$$\begin{aligned} \Lambda_{tr}^* &= \frac{S\{\boldsymbol{q}^*|\boldsymbol{z}\}}{S\{\boldsymbol{q}_r|\boldsymbol{z}\}} = \frac{S\{\boldsymbol{q}_t^{(l)}|\boldsymbol{z}\}}{S\{\boldsymbol{q}_t^{(0)}|\boldsymbol{z}\}} = \prod_{i=1}^{l} \frac{S\{\boldsymbol{q}_t^{(i)}|\boldsymbol{z}\}}{S\{\boldsymbol{q}_t^{(i-1)}|\boldsymbol{z}\}} \\ &= \prod_{i=1}^{l} \Lambda_{tr}(k_i, \boldsymbol{q}_t^{(i-1)}) = \prod_{i=1}^{l} \Lambda_{tr}(k_i, \boldsymbol{q}_r) > 1 \end{aligned}$$

which is (13). Note that whenever the test sequence $\boldsymbol{q}_r$ is highly populated at a certain iteration, the set B_1 may be a null set. Then only a single spike replacement $(l = 1)$ can occur at this iteration.

Property 2: At each iteration, the MMLR detection algorithm searches for the SMLR $\boldsymbol{q}$ sequence [$q_t^{(1)}(k)$, see (16)] at the first recursion. In other words, the SMLR $\boldsymbol{q}$ sequence is used as the initial sequence for the following recursive search for the MMLR $\boldsymbol{q}$ sequence. Therefore, when the number of changes in spike location at each iteration is artificially fixed to be unity, the MMLR detector reduces to the SMLR detector.

Property 3: At each iteration, the MMLR detector changes at most $l_{\max} = [(N - 1)/M] + 1$ spikes where $[x]$ denotes the integer part of x.

Proof: Let $\{j_1, j_2, \cdots, j_l\} = \{k_1, k_2, \cdots, k_l\}$ with $j_i < j_{i+1}$ for all i. The maximum value $l_{\max}$ of l occurs when $j_{i+1} - j_i$

$= M$, $j_l + M > N$ and $j_1 - M < 1$. It is easy to see that $l_{\max} = [(N-1)/M] + 1$.

Property 4: For the case $M > N - 1$ ($l_{\max} = 1$), the MMLR detector is equivalent to the SMLR detector.

Property 5: The same amount of computation, an optimal smoother for computing $\Lambda_{tr}(k, \boldsymbol{q}_r)$, must be spent for each iteration for both the MMLR and SMLR detectors.

Although there are many possibilities that the number of spike replacement l might be only 1 for some iterations before convergence, the MMLR detection algorithm is much faster than the SMLR detection algorithm for the case that $N \gg M$ and λ is small, i.e., $q(k)$ is a sparse spike train. This case is a general case in seismology.

IV. Computer Simulations

In this section, we discuss two examples which illustrate the performance of the proposed MMLR detector for the case when the $\boldsymbol{q}$ sequence is a sparse sequence. In both examples, we generated a Bernoulli–Guassian sequence $\mu(k)$ with known parameters λ and C. $\mu(k)$ was convolved with a known wavelet $v(k)$ to which white Gaussian noise with known variance R was added to produce the synthetic data.

For the first example, $\lambda = 0.04$, $C = 0.0225$, $R = 1.5503 \times 10^{-4}$, and $N = 400$. The impulse response $v(k)$ of the signal distorting system taken from [3] was a long wavelet with a broad frequency band (see [8]) and length $M \cong 80$. $q(k) = 0$ for all k was used to be the initial $\boldsymbol{q}$ sequence for both MMLR and SMLR detectors.

For the second example, λ, C, N, $\mu(k)$, and initial $\boldsymbol{q}$ sequence are the same as those used in the first example, but $R = 1.2291 \times 10^{-7}$ and $v(k)$ was taken from [6]. The length of this wavelet is about half of that in the first example. This wavelet is a narrowband wavelet. Some other simulation results were given in [9] in more detail.

For the first example, the MMLR and SMLR detection algorithms yielded the same performance (identical results). For the second example, the MMLR and SMLR detection algorithms yielded comparable performance. For the first example, the SMLR detector spent 14 iterations, but the MMLR detection algorithm spent only 5 iterations for convergence. For the second example, the SMLR detection algorithm spent 28 iterations, but the MMLR detection algorithm spent only 10 iterations. In other words, for these two examples, the MMLR detection algorithm was about three times faster than the SMLR detection algorithm and yielded comparable performance.

V. Summary and Conclusions

In this correspondence, we proposed a new iterative maximum likelihood detection algorithm, the MMLR detector, which is computationally faster than the well-known SMLR detector, for Bernoulli–Gaussian processes. An initial $\boldsymbol{q}$ sequence is needed for both detectors. Both detection algorithms are suboptimal maximum likelihood detectors and yield comparable performance. At each iteration, $l \geq 1$ spike locations are updated such that the likelihood function is increased. The value of l varies from iteration to iteration, but the computational effort is equivalent to an optimal smoother no matter what the value of l is. Although there are cases that the MMLR detector reduces to the SMLR detector, the MMLR detection algorithm is much faster than the SMLR detection algorithm for the case that $N \gg M$ and λ is small, i.e., $q(k)$ is a sparse spike sequence. This case is a general case in seismic applications. We discussed two simulation examples for this case.

The strategy of the proposed MMLR detector for changing a spike location during each iteration may not be the most efficient one. Other strategies may improve the speed of the MMLR detector. We leave this task for future research.

Appendix A
Proof of Theorem 1

Proof: Kormylo and Mendel [3] showed that

$$\ln \Lambda_{tr}(k, \boldsymbol{q}_r) = \frac{1}{2}\frac{Cf_k^2\rho_k}{1 + Ca_k\rho_k} - \frac{1}{2}\ln(1 + Ca_k\rho_k) + \rho_k \ln\frac{\lambda}{1-\lambda} \tag{A1}$$

where

$$f_k = \boldsymbol{v}_k'\Omega^{-1}\boldsymbol{z}, \tag{A2}$$

$$a_k = \boldsymbol{v}_k'\Omega^{-1}\boldsymbol{v}_k, \tag{A3}$$

$$\rho_k = q_t(k) - q_r(k), \tag{A4}$$

$$\Omega = E[\boldsymbol{z}\boldsymbol{z}' \mid \boldsymbol{q} = \boldsymbol{q}_r] = CVQ_rV' + RI \tag{A5}$$

$$\boldsymbol{z} = \text{col}\,(z(1), z(2), \cdots, z(N)), \tag{A6}$$

$$Q_r = \text{diag}\,(q_r(1), q_r(2), \cdots, q_r(N)), \tag{A7}$$

$$V = \begin{bmatrix} v(0) & 0 & \cdots & 0 \\ v(1) & v(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v(N-1) & v(N-2) & \cdots & v(0) \end{bmatrix} \tag{A8}$$

and $\boldsymbol{v}_k$ is the kth column of V.

From (A1), (A2), and (A3), we see that only the following two equations need to be shown in order to show (15). These equations are

$$\boldsymbol{v}_k'\Omega_1^{-1}\boldsymbol{z} = \boldsymbol{v}_k'\Omega_2^{-1}\boldsymbol{z}, \ \forall k \notin D \tag{A9}$$

and

$$\boldsymbol{v}_k'\Omega_1^{-1}\boldsymbol{v}_k = \boldsymbol{v}_k'\Omega_2^{-1}\boldsymbol{v}_k, \ \forall k \notin D \tag{A10}$$

where $\Omega_1 = E[\boldsymbol{z}\boldsymbol{z}' \mid \boldsymbol{q} = \boldsymbol{q}_r^{(1)}]$ and $\Omega_2 = E[\boldsymbol{z}\boldsymbol{z}' \mid \boldsymbol{q} = \boldsymbol{q}_r^{(2)}]$. Next, we show (A9) and (A10).

Because

$$\begin{aligned}\Omega_2 &= C\sum_{i=1}^{N} q_r^{(2)}(i)\,\boldsymbol{v}_i\boldsymbol{v}_i' + RI \\ &= C\sum_{i=1}^{N} q_r^{(1)}(i)\,\boldsymbol{v}_i\boldsymbol{v}_i' + RI \\ &\quad + C\left(q_r^{(2)}(m) - q_r^{(1)}(m)\right)\boldsymbol{v}_m\boldsymbol{v}_m' \\ &= \Omega_1 + Cd_m\boldsymbol{v}_m\boldsymbol{v}_m' \end{aligned} \tag{A11}$$

where

$$d_m = q_r^{(2)}(m) - q_r^{(1)}(m). \tag{A12}$$

By matrix inversion identity, we have

$$\Omega_2^{-1} = \Omega_1^{-1} - \frac{Cd_m\Omega_1^{-1}\boldsymbol{v}_m\boldsymbol{v}_m'\Omega_1^{-1}}{1 + Cd_m\boldsymbol{v}_m'\Omega_1^{-1}\boldsymbol{v}_m}. \tag{A13}$$

Thus,

$$\boldsymbol{v}_k'\Omega_2^{-1}\boldsymbol{z} = \boldsymbol{v}_k'\Omega_1^{-1}\boldsymbol{z} - \frac{Cd_m(\boldsymbol{v}_k\Omega_1^{-1}\boldsymbol{v}_m)(\boldsymbol{v}_m'\Omega_1^{-1}\boldsymbol{z})}{1 + Cd_m\boldsymbol{v}_m'\Omega_1^{-1}\boldsymbol{v}_m} \tag{A14}$$

and

$$\boldsymbol{v}_k'\Omega_2^{-1}\boldsymbol{v}_k = \boldsymbol{v}_k'\Omega_1^{-1}\boldsymbol{v}_k - \frac{Cd_m(\boldsymbol{v}_k\Omega_1^{-1}\boldsymbol{v}_m)^2}{1 + Cd_m\boldsymbol{v}_m'\Omega_1^{-1}\boldsymbol{v}_m}. \tag{A15}$$

In Appendix B, we show that for a sequence $\boldsymbol{q} \in \Gamma$ and $m \in [n_1, n_2]$, the associated covariance matrix Ω [see (A5)] satisfies the following equation:

$$\boldsymbol{v}_k'\Omega^{-1}\boldsymbol{v}_m = 0 \ \forall k \notin D. \tag{A16}$$

Therefore, the second terms of (A14) and (A15) are zero for $k \notin A$ and $m \in [n_1, n_2]$, and this proves (A9) and (A10) as required.

Appendix B
Proof of (A16)

Proof: For a sequence $\boldsymbol{q} \in \Gamma$ from (A5) we have

$$\Omega = CVQV' + RI = C \sum_{i=1}^{N} q(i) v_i v_i' + RI = J_1 + J_2 + J_3 + RI \tag{B1}$$

where

$$J_1 = C \sum_{i=1}^{n_1 - M} q(i) v_i v_i', \tag{B2}$$

$$J_2 = C \sum_{i=n_1}^{n_2} q(i) v_i v_i', \tag{B3}$$

and

$$J_3 = C \sum_{i=n_2+M}^{N} q(i) v_i v_i'. \tag{B4}$$

Let us simplify J_1, J_2, and J_3, respectively. From (B2), we see that

$$J_1 = \begin{pmatrix} CV_1Q_1V_1' & 0 \\ 0 & 0 \end{pmatrix} \tag{B5}$$

where V_1 is an $(n_1 - 1) \times (n_1 - M)$ matrix and

$$V_1 = \begin{bmatrix} v(0) & 0 & \cdots & 0 \\ v(1) & v(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v(M-1) & v(M-2) & \cdots & v(0) \\ 0 & v(M-1) & \cdots & v(1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v(M-1) \end{bmatrix} \tag{B6}$$

and

$$Q_1 = \text{diag}\,(q(1), q(2), \cdots, q(n_1 - M)). \tag{B7}$$

From (B3), we can see that

$$J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & CV_2Q_2V_2' & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{B8}$$

where V_2 is an $(n_2 - n_1 + M) \times (n_2 - n_1 + 1)$ matrix with the same form as V_1 and

$$Q_2 = \text{diag}\,(q(n_1), q(n_1 + 1), \cdots, q(n_2)). \tag{B9}$$

Similarly, we have

$$J_3 = \begin{pmatrix} 0 & 0 \\ 0 & CV_3Q_3V_3' \end{pmatrix} \tag{B10}$$

where V_3 is an $(N - n_2 - M + 1) \times (N - n_2 - M + 1)$ matrix with the same form as V_1 and

$$Q_3 = \text{diag}\,(q(n_2 + M), q(n_2 + M + 1), \cdots, q(N)). \tag{B11}$$

Substituting (B5), (B8), and (B10) into (B1) gives

$$\Omega = \begin{bmatrix} CV_1Q_1V_1' + RI_1 & 0 & 0 \\ 0 & CV_2Q_2V_2' + RI_2 & 0 \\ 0 & 0 & CV_3Q_3V_3' + RI_3 \end{bmatrix} \tag{B12}$$

where I_1, I_2, and I_3 are identity matrices with dimensions $(n_1 - 1) \times (n_1 - 1)$, $(n_2 - n_1 + M) \times (n_2 - n_1 + M)$, and $(N - n_2 - M + 1) \times (N - n_2 - M + 1)$, respectively. Thus,

$$\Omega^{-1} = \begin{bmatrix} (CV_1Q_1V_1' + RI_1)^{-1} & 0 & 0 \\ 0 & (CV_2Q_2V_2' + RI_2)^{-1} & 0 \\ 0 & 0 & (CV_3Q_3V_3' + RI_3)^{-1} \end{bmatrix}. \tag{B13}$$

Directly computing $v_k' \Omega^{-1} v_m$ with $m \in [n_1, n_2]$ using (B13) proves that (A16) holds.

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Statistical Performance of Single Sinusoid Frequency Estimation in White Noise Using State-Variable Balancing and Linear Prediction

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***Abstract*—This correspondence presents a statistical analysis of frequency estimation using state-variable balancing for a single sinusoid in the presence of additive noise at high signal-to-noise ratios. The calculated variance is compared to the performance of the frequency es-**

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